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Finite-dimensional representations of the quantum group $GL_{p,q}(2)$ using the exponential map from $U_{p,q}(gl(2))$

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Abstract. Using the Fronsdal–Galindo formula for the exponential mapping from the quantum algebra $U_{p,q}(gl(2))$ to the quantum group $GL_{p,q}(2)$, we show how the $(2j+1)$-dimensional representations of $GL_{p,q}(2)$ can be obtained by 'exponentiating' the well known $(2j+1)$-dimensional representations of $U_{p,q}(gl(2))$ for $j = 1, \frac{3}{2}, \ldots$; $j = \frac{1}{2}$ corresponds to the defining two-dimensional T-matrix. The earlier results on the finite-dimensional representations of $GL_q(2)$ and $SL_q(2)$ (or $SU_q(2)$) are obtained when $p = q$. Representations of $U_{\overline{q},q}(2)(q \in \mathbb{C}\backslash\mathbb{R}$ and $U_q(2)(q \in \mathbb{R}\backslash\{0\})$ are also considered. The structure of the Clebsch–Gordan matrix for $U_{p,q}(gl(2))$ is studied. The same Clebsch–Gordan coefficients are applicable in the reduction of the direct product representations of the quantum group $GL_{p,q}(2)$.

1. Introduction

The analysis of the bialgebraic duality relationship between the pair of Hopf algebras $\mathcal{A} = A\left(GL_{p,q}(2)\right)$ (or $Fun_{p,q}(GL(2))$) and $\mathcal{U} = U_{p,q}(gl(2))$ by Fronsdal and Galindo [1] provides the first example of generalization of the exponential relationship, obtained between the classical Lie groups and algebras, through the construction of the form of such a mapping from $U_{p,q}(gl(2))$ to the quantum group $GL_{p,q}(2)$. Following [1], Bonechi *et al* [2] studied the forms of the exponential relationship for a few other examples of quantum groups, namely the quantized Heisenberg, Euclidean and Galilei groups. More recently, Morozov and Vinet [3] have obtained a generalization of the result of [1] for any simple quantum group with a single deformation parameter. For $GL_q(2)$, Finkelstein [4] has obtained the representations using theory of invariants and studied the converse map $GL_q(2) \longrightarrow U_q(gl(2))$ leading to the identification of the generators of $U_q(gl(2))$ as the infinitesimal generators of $GL_q(2)$: hence, the representations of $U_q(gl(2))$ are also derived.

Here, we demonstrate explicitly how the Fronsdal–Galindo theory leads to a straightforward derivation of the finite-dimensional representaions of $GL_{p,q}(2)$. The earlier results on the finite-dimensional irrreducible representaions of $GL_q(2)$ [4] and $SL_q(2)$ (or $SU_q(2)$) [5–11] leading to the q-analogues of the classical Wigner d-functions or spherical functions, are seen to follow in the special case when $p = q$. We consider briefly also the quantum groups $U_{\overline{q},q}(2)(q \in \mathbb{C}\backslash\mathbb{R})$ and $U_q(2)(q \in \mathbb{R}\backslash\{0\})$.

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The question of q-analogues of the Clebsch–Gordan coefficients (CGCs) arises naturally when one considers the representation theory of quantum groups. The CGCs for $U_q(su(2))$ and $SU_q(2)$ have been discussed extensively (see [7, 8, 10, 11]). In the case of $GL_{p,q}(2)$ [12–15], the structure of its dual Hopf algebra, $U_{p,q}(gl(2))$, is known from studies using different approaches [16–18]. It is clear that one can have only $U_{p,q}(u(2))$ non-trivially and any (p, q)-deformation of $U(su(2))$ would depend effectively only on a single parameter ($\sqrt{pq}$). So, the earlier studies on a (p, q)-deformed $su(2)$ algebra led naturally to the conclusion that the corresponding deformed CGCs depend only on $\sqrt{pq}$ (see [19, 20]). Here, we shall arrive at the structure of the Clebsch–Gordan matrix for $U_{p,q}(gl(2))$ based on Reshetikhin's general theory of quantum algebras with multiple deformation parameters [21]. The same Clebsch–Gordan matrices can be used for the reduction of direct product representations of the quantum group $GL_{p,q}(2)$.

Before proceeding to consider the representation theory of $GL_{p,q}(2)$, in particular, let us first recall some elements of the general theory of representations of quantum groups. The Hopf algebra $A(G_q)$, the algebra of functions on a quantum group G_q, has a non-Abelian coordinate ring generated by the (variable) elements of a T-matrix satisfying the RTT-relation

$$RT_1T_2 = T_2T_1R \qquad \text{with} \qquad T_1 = T \otimes \mathbf{1} \quad T_2 = \mathbf{1} \otimes T \tag{1.1}$$

where the R-matrix is a solution of the Yang–Baxter equation. The coproduct maps of the elements of T, $\{T_{mk}\}$, specified by

$$\Delta\,(T_{mk}) = \sum_j T_{mj} \otimes T_{jk} \qquad \text{or} \qquad \Delta(T) = T\dot{\otimes}T \tag{1.2}$$

provide a unique homomorphism of the $\mathbb{C}$-algebra generated by $\{T_{mk}\}$ subject to the relations (1.1). One may choose a basis for $A(G_q)$ with any convenient parametrization and take the required coproduct maps as induced from (1.2). A $\mathbb{C}$-vector subspace V spanned by $\{V_m | m = 1, 2, \ldots, n\}$ of $A(G_q)$ is said to carry an n-dimensional representation of G_q if it forms a left subcomodule of A such that

$$\Delta(V) = \mathcal{T}\dot{\otimes}V \qquad \text{or} \qquad \Delta\,(V_m) = \sum_{k=1}^{n} \mathcal{T}_{mk} \otimes V_k \quad \forall\, m = 1, 2, \ldots, n \tag{1.3}$$

where $\{\mathcal{T}_{mk} \in A\}$ are called the elements of the 'representation matrix' $\mathcal{T}$. Then, the comodule structure of $\{V_m\}$ implies, as in (1.2),

$$\Delta(\mathcal{T}_{mk}) = \sum_{j=1}^{n} \mathcal{T}_{mj} \otimes \mathcal{T}_{jk} \qquad \text{i.e. } \Delta(\mathcal{T}) = \mathcal{T}\dot{\otimes}\mathcal{T}\,. \tag{1.4}$$

For example, the elements of the first (or, any) column of the T-matrix constitute such a $\mathbb{C}$-vector subspace of $A(G_q)$ carrying the defining representation with $\mathcal{T} = T$. Essentially, a representation matrix $\mathcal{T}$ of G_q is a generalization of the T-matrix, with $\{\mathcal{T}_{mk} \in A(G_q)\}$ and with relation (1.4) satisfied.

The paper is organized as follows. In section 2, we recall briefly the main properties of the quantum group $GL_{p,q}(2)$ and its dual Hopf algebra $U_{p,q}(gl(2))$. In section 3, we describe the Fronsdal–Galindo method for exponentiating the representations of $U_{p,q}(gl(2))$ to obtain the representations of $GL_{p,q}(2)$. Section 4 gives the explicit representation matrices $\{\mathcal{T}\}$ for $GL_{p,q}(2)$ and shows how the earlier results on the representations of $GL_q(2)$, $SL_q(2)$ and $SU_q(2)$ are obtained in the appropriate limits. In section 5 we briefly remark on the quantum group $U_{\bar{q},q}(2)$. Section 6 presents the solution to the problem of CGCs for $U_{p,q}(gl(2))$ and $GL_{p,q}(2)$. Section 7 contains several concluding remarks.

2. The properties of $GL_{p,q}(2)$ and $U_{p,q}(gl(2))$

For the quantum group $GL_{p,q}(2)$ the defining T-matrix, or the defining representation matrix, is specified by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

with the commutation relations

$$\begin{array}{llll} ab = qba & cd = qdc & ac = pca & bd = pdb \\ bc = (p/q)cb & ad - da = (q - p^{-1})bc & & \end{array} \tag{2.2}$$

following from the RTT-relation (1.1) corresponding to

$$R = Q^{\frac{1}{2}} \begin{pmatrix} Q^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & Q^{-1} - Q & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & Q^{-1} \end{pmatrix} \tag{2.3}$$

where

$$Q = \sqrt{pq} \qquad \lambda = \sqrt{p/q}\,. \tag{2.4}$$

Here we are concerned only with the case of generic values of Q, $\lambda \in \mathbb{C}\backslash\{0\}$. Also, it may be noted that we shall use both sets of notation (Q, λ) and (p, q) for the deformation parameters interchangably, according to convenience, with the relationship (2.4) always implied. The coproduct maps (1.2) are explicitly given by

$$\Delta(T) = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}. \tag{2.5}$$

The quantum determinant of T defined by

$$\mathcal{D} = ad - qbc = ad - pcb \tag{2.6}$$

satisfies the commutation relations

$$\mathcal{D}a = a\mathcal{D} \qquad \mathcal{D}b = \lambda^{-2}b\mathcal{D} \qquad \mathcal{D}c = \lambda^{2}c\mathcal{D} \qquad \mathcal{D}d = d\mathcal{D} \tag{2.7}$$

and is a group-like element such that

$$\Delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}\,. \tag{2.8}$$

The algebra $\mathcal{U}$ dual to $\mathcal{A} = A(GL_{p,q}(2))$, namely $U_{p,q}(gl(2))$, may be presented in a standard form as follows. The generators $\{J_0, J_\pm, Z\}$ can be taken to satisfy the algebra

$$\begin{array}{lll} [J_0, J_\pm] & = & \pm J_\pm \qquad [J_+, J_-] = [2J_0]_Q \\ [Z, J_0] & = & 0 \qquad [Z, J_\pm] = 0 \end{array} \tag{2.9}$$

with

$$[X]_{\mathbf{q}} = \frac{\mathbf{q}^X - \mathbf{q}^{-X}}{\mathbf{q} - \mathbf{q}^{-1}} \qquad \forall\, \mathbf{q}, X\,. \tag{2.10}$$

Whereas the algebraic relations (2.9) do not depend on λ, the coproducts do and are given by, up to equivalence,

$$\begin{array}{ll} \Delta(J_\pm) = & J_\pm \otimes Q^{-J_0}\lambda^{\pm Z} + Q^{J_0}\lambda^{\mp Z} \otimes J_\pm \\ \Delta(J_0) = & J_0 \otimes \mathbb{1} + \mathbb{1} \otimes J_0 \qquad \Delta(Z) = Z \otimes \mathbb{1} + \mathbb{1} \otimes Z\,. \end{array} \tag{2.11}$$

The associated universal $\mathcal{R}$-matrix, relating the coproduct Δ and the opposite coproduct $\Delta' = \sigma\,\Delta$, with $\sigma\,(u \otimes v) = v \otimes u, \quad \forall\, u, v \in \mathcal{U}$, through the equation

$$\Delta'(u) = \mathcal{R}\Delta(u)\mathcal{R}^{-1} \qquad \forall\, u \in \mathcal{U} \tag{2.12}$$

is

$$\mathcal{R} = Q^{-2(J_0\otimes J_0)}\lambda^{2(Z\otimes J_0 - J_0\otimes Z)} \sum_{n=0}^{\infty} \frac{(1-Q^2)^n}{[n]!}\, Q^{-\frac{1}{2}n(n-1)}(Q^{-J_0}\lambda^Z J_+ \otimes Q^{J_0}\lambda^Z J_-)^n \tag{2.13}$$

where

$$[n] = [n]_Q \qquad [n]! = [n][n-1]\cdots[2][1] \qquad [0]! = 1\,. \tag{2.14}$$

Hereafter, unless otherwise specified, $[n] = [n]_Q$. The $\mathcal{R}$-matrix (2.13), say $\mathcal{R}_{Q,\lambda}$, is easily obtained [22] following the observation that, if we denote by $\Delta_{Q,\lambda}$ the coproduct defined by (2.11), then,

$$\Delta_{Q,\lambda}(u) = F\Delta_{Q,\lambda=1}(u)F^{-1} \quad \forall\, u \in \mathcal{U} \qquad \text{with } F = \lambda^{(J_0\otimes Z - Z\otimes J_0)} \tag{2.15}$$

and by Reshetikhin's theory [21]

$$\mathcal{R}_{Q,\lambda} = F^{-1}\mathcal{R}_{Q,\lambda=1}F^{-1}\,. \tag{2.16}$$

We are not concerned here with the other aspects of the Hopf algebraic structure, namely the counit and the antipode.

3. The exponential map from $U_{p,q}(gl(2))$ to $GL_{p,q}(2)$

Let us now describe the exponential map from $U_{p,q}(gl(2))$ to $GL_{p,q}(2)$, *à la* Fronsdal and Galindo [1]. To this end, we have to redefine the generators of $U_{p,q}(gl(2))$ as

$$\hat{J}_+ = J_+Q^{-(J_0+\frac{1}{2})}\lambda^{Z-\frac{1}{2}} \qquad \hat{J}_- = Q^{(J_0+\frac{1}{2})}\lambda^{Z-\frac{1}{2}}J_- \qquad \hat{J}_0 = J_0 \qquad \hat{Z} = Z \tag{3.1}$$

with the algebra

$$\begin{aligned} &[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm \qquad [\hat{J}_+, \hat{J}_-] = \lambda^{2\hat{Z}-1}[2\hat{J}_0] \\ &[\hat{Z}, \hat{J}_0] = 0 \qquad [\hat{Z}, \hat{J}_\pm] = 0 \end{aligned} \tag{3.2}$$

and the induced coproducts

$$\begin{aligned} &\Delta(\hat{J}_+) = \hat{J}_+ \otimes Q^{-2\hat{J}_0}\lambda^{2\hat{Z}} + \mathbb{1}\otimes\hat{J}_+ \qquad \Delta(\hat{J}_-) = \hat{J}_-\otimes\mathbb{1} + Q^{2\hat{J}_0}\lambda^{2\hat{Z}}\otimes\hat{J}_- \\ &\Delta(\hat{J}_0) = \hat{J}_0\otimes\mathbb{1} + \mathbb{1}\otimes\hat{J}_0 \qquad \Delta(\hat{Z}) = \hat{Z}\otimes\mathbb{1} + \mathbb{1}\otimes\hat{Z}\,. \end{aligned} \tag{3.3}$$

Let the element 'a' of T be taken to be invertible [1] and expressed as

$$a = \mathrm{e}^{\alpha}\,. \tag{3.4}$$

Since we are dealing only with non-singular T-matrices, we shall represent

$$\mathcal{D} = \mathrm{e}^{-2\phi} = \xi^{-2}\,. \tag{3.5}$$

Further, let us take

$$\beta = a^{-1}b \qquad \gamma = ca^{-1} \qquad \delta = \alpha + 2\phi\,. \tag{3.6}$$

The variables $\{a, b, c, d\}$ can be expressed as

$$a = \mathrm{e}^{\alpha} \qquad b = \mathrm{e}^{\alpha}\beta \qquad c = \gamma\mathrm{e}^{\alpha} \qquad d = \gamma\mathrm{e}^{\alpha}\beta + \mathrm{e}^{-\delta}\,. \tag{3.7}$$

The set of new variables $\{\alpha, \beta, \gamma, \delta\}$ are seen to form a Lie algebra:

$$\left.\begin{array}{ll} [\alpha,\beta]=(\rho-\theta)\beta & [\alpha,\gamma]=(\rho+\theta)\gamma \\ [\delta,\beta]=(\rho+\theta)\beta & [\delta,\gamma]=(\rho-\theta)\gamma \\ [\alpha,\delta]=0 \quad [\beta,\gamma]=0 & \end{array}\right\} \quad \text{with } Q=\mathrm{e}^{\rho} \quad \lambda=\mathrm{e}^{\theta}. \tag{3.8}$$

Then, following Fronsdal and Galindo [1], one can write down a 'universal $\mathcal{T}$-matrix' for $GL_{p,q}(2)$ as

$$\mathcal{T}=\mathcal{E}xp_{Q^{-2}}\{\gamma\hat{J}_-\}\exp\{\alpha(\hat{J}_0+\hat{Z})+\delta(\hat{J}_0-\hat{Z})\}\mathcal{E}xp_{Q^2}\{\beta\hat{J}_+\} \tag{3.9}$$

where

$$\mathcal{E}xp_{\mathbf{q}^2}\{X\}=\sum_{n=0}^{\infty}\left\{\prod_{k=1}^{n}(\mathbf{q}^{2k}-1)\right\}^{-1}(\mathbf{q}^2-1)^n X^n=\sum_{n=0}^{\infty}\frac{\mathbf{q}^{-\frac{1}{2}n(n-1)}}{[n]_{\mathbf{q}}!}X^n. \tag{3.10}$$

The universal $\mathcal{T}$-matrix (3.9) embodies all the finite-dimensional representations of $GL_{p,q}(2)$: substituting the finite-dimensional numerical matrices representing $\{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}$ in the expression (3.9) for $\mathcal{T}$, expanding it, and re-expressing the resulting matrix elements in terms of $\{a, b, c, d\}$ and $\xi=\mathcal{D}^{-\frac{1}{2}}$ using the relations (2.6), (3.4)–(3.8), one obtains all the finite-dimensional representation matrices $\{\mathcal{T}\}$ of $GL_{p,q}(2)$ in the sense of the representation theory outlined in the introduction. Though the definition of the matrix elements of $\mathcal{T}$ by (3.9) involves a^{-1}, it is found to be possible to express them completely in terms of $\{a, b, c, d\}$ and the group-like ξ (with $\Delta(\xi)=\xi\otimes\xi$), using the relations (2.6), (3.4)–(3.8).

Except for the additional central element Z, the algebra (2.9) is the same as the standard $U_Q(sl(2))$ for which all the finite-dimensional irreducible representations are known (see [23, 24]): for generic values of $Q\in\mathbb{C}\backslash\{0\}$, these are the straightforward Q-analogues of the $(2j+1)$-dimensional (spin-j) representations of the classical $sl(2)$, with $j=0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. So, with the extra central element as $Z=z\mathbf{1}$, the $(2j+1)$-dimensional irreducible representations of $\{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}$ obeying (3.2), say $\{\Gamma^{(\mu)} \mid \mu=(j,z)\}$, are readily obtained (see also [25] for more details on the dual Hopf algebras $A(GL_{p,q}(2))$ and $U_{p,q}(gl(2))$ and representation theory of quantized universal enveloping algebras $\{U_q(G)\}$ in general). Let us call the matrix obtained by substituting the representation $\Gamma^{(\mu)}$ in the expression (3.9) for $\mathcal{T}$ as $\mathcal{T}^{(\mu)}$. The matrix elements are labelled as $\{\mathcal{T}^{(\mu)}_{mk} \mid m, k=j, j-1, \ldots, -(j-1), -j\}$.

The basic theory underlying the Fronsdal–Galindo formalism is as follows. Let $\{x^A\}$ be a basis for $A(G_q)$ such that

$$x^Bx^C=\sum_A h^{BC}_A x^A \qquad \Delta(x^C)=\sum_{A,B} f^C_{AB}x^A\otimes x^B. \tag{3.11}$$

Let $\{X_A\}$ be the basis of $U_q(G)$ chosen such that

$$X_AX_B=\sum_C f^C_{AB}X_C \qquad \Delta(X_A)=\sum_{B,C} h^{BC}_A X_B\otimes X_C. \tag{3.12}$$

Then, the universal $\mathcal{T}$-matrix defined by

$$\mathcal{T}=\sum_A x^AX_A \tag{3.13}$$

satisfies the equation

$$\Delta(\mathcal{T})=\sum_A \Delta(x^A)X_A=\mathcal{T}\dot{\otimes}\mathcal{T} \tag{3.14}$$

in view of the duality relations (equations (3.11), (3.12)). Equation (3.13) should be interpreted as defining a universal $\mathcal{T}$-matrix in the sense that $\mathcal{T}^{(\mu)}=\sum_A x^AX^{(\mu)}_A\in A(G_q)$, for $\{X_A^{(\mu)}\}$, the numerical representations of $\{X_A\}$, in any representation $\Gamma^{(\mu)}$. Thus, the elements $\{\mathcal{T}_{mk}^{(\mu)} \in A(G_q)\}$ form a representation matrix for the quantum group G_q and the formula (3.13) expresses abstractly the exponential map $U_q(G) \longrightarrow G_q$. In the present context of $GL_{p,q}(2)$, the basis elements of $A(GL_{p,q}(2))$ and $U_{p,q}(gl(2))$ are given (see [1] for details of derivation), respectively, by

$$\left.\begin{array}{l} x^A = \gamma^{a_1}\alpha^{a_2}\delta^{a_3}\beta^{a_4} \\ X_A = \dfrac{Q^{\frac{1}{2}a_1(a_1-1)}\hat{J}_-^{a_1}}{[a_1]!}\dfrac{(\hat{J}_0+\hat{Z})^{a_2}}{a_2!}\dfrac{(\hat{J}_0-\hat{Z})^{a_3}}{a_3!}\dfrac{Q^{-\frac{1}{2}a_4(a_4-1)}\hat{J}_+^{a_4}}{[a_4]!} \end{array}\right\} \qquad a_1, a_2, a_3, a_4 = 0, 1, 2, \ldots . \tag{3.15}$$

As was noted by Bonechi *et al* [2], the relations (3.11), (3.12) also imply

$$\mathcal{T}_1\mathcal{T}_2 = \sum_A x^A \Delta(X_A) \qquad \mathcal{T}_2\mathcal{T}_1 = \sum_A x^A \Delta'(X_A) \tag{3.16}$$

where $\mathcal{T}_1 = \mathcal{T} \otimes \mathbb{1}$ and $\mathcal{T}_2 = \mathbb{1} \otimes \mathcal{T}$. Hence, in the present case one would have

$$R^{(\mu\otimes\mu)}\mathcal{T}_1^{(\mu)}\mathcal{T}_2^{(\mu)} = \mathcal{T}_2^{(\mu)}\mathcal{T}_1^{(\mu)}R^{(\mu\otimes\mu)} \tag{3.17}$$

in accordance with the FRT formalism of quantum groups [26] where $R^{(\mu\otimes\mu)}$ is the R-matrix corresponding to the direct product of two representations $(\Gamma^{(\mu)})$ and obtained by substituting the respective representations in the formula (2.13) for the universal $\mathcal{R}$.

Now, we note that the R-matrix (2.3) defining the T-matrix (2.1), (2.2) is $R^{(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})}$. For a generic value of z $(\neq \frac{1}{2})$, $R^{(\frac{1}{2},z)\otimes(\frac{1}{2},z)}$ has the same form as (2.3) with λ replaced by λ^{2z} and thus defines a T-matrix whose elements will obey the commutation relations of the same form as (2.2) with p and q replaced, respectively, by $p' = p^{z+\frac{1}{2}}/q^{z-\frac{1}{2}}$ and $q' = q^{z+\frac{1}{2}}/p^{z-\frac{1}{2}}$. In general, it is clear from (2.15) and (2.16) that $R^{(j,z)\otimes(j,z)}$ would have the same form as $R^{(j,\frac{1}{2})\otimes(j,\frac{1}{2})}$ with λ replaced by λ^{2z}.

4. Explicit representation matrices $\{\mathcal{T}^{(j,z)}\}$

First, let us consider the matrices $\mathcal{T}^{(\frac{1}{2},\frac{1}{2})}$ and $\mathcal{T}^{(1,\frac{1}{2})}$ explicitly and, then, generalize the result. To verify that $\mathcal{T}^{(\frac{1}{2},\frac{1}{2})}$ is the defining T-matrix (2.1) we have to substitute the matrices

$$\hat{J}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{J}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{J}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{Z} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1}$$

in (3.9) and expand. The result is

$$\begin{aligned} \mathcal{T}^{(\frac{1}{2},\frac{1}{2})} &= \mathcal{E}xp_{Q^{-2}}\left\{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\right\}\exp\left\{\begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}\right\}\mathcal{E}xp_{Q^2}\left\{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right\} \\ &= \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\delta} \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\alpha} & e^{\alpha}\beta \\ \gamma e^{\alpha} & \gamma e^{\alpha}\beta + e^{-\delta} \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & ca^{-1}b + a^{-1}\mathcal{D} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned} \tag{4.2}$$

as expected; this is, in fact, the starting point of the Fronsdal–Galindo formalism. It may be noted that at the level of the two-dimensional defining representation $\mathcal{E}xp\{\ \}$ can be replaced by ordinary $\exp\{\ \}$ (see Finkelstein [4] and Akulov *et al* [27] for the use of such a realization of the 2×2 T-matrix for $GL_q(2)$ and $SL_q(2)$). The deformed structure of the exponential map (3.9), brought out by the Fronsdal–Galindo approach, is revealed only in dimensions > 2.

For $\Gamma^{(1,\frac{1}{2})}$, with $[2] = Q + Q^{-1}$,

$$\hat{J}_+ = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & Q^{-\frac{1}{2}} & 0 \\ 0 & 0 & Q^{\frac{1}{2}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{J}_- = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 \\ Q^{\frac{1}{2}} & 0 & 0 \\ 0 & Q^{-\frac{1}{2}} & 0 \end{pmatrix}$$
$$\hat{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{Z} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Substituting this representation in (3.9) we obtain

$$\mathcal{T}^{(1,\frac{1}{2})} = \begin{pmatrix} e^{\phi+2\alpha} & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} e^{\phi+2\alpha}\beta & Q^{-1} e^{\phi+2\alpha}\beta^2 \\ [2]^{\frac{1}{2}} Q^{\frac{1}{2}} \gamma e^{\phi+2\alpha} & [2]\gamma e^{\phi+2\alpha}\beta + e^{-\phi} & \begin{array}{c}([2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \gamma e^{\phi+2\alpha}\beta^2 \\ +[2]^{\frac{1}{2}} Q^{\frac{1}{2}} e^{-\phi}\beta)\end{array} \\ Q\gamma^2 e^{\phi+2\alpha} & \begin{array}{c}([2]^{\frac{1}{2}} Q^{\frac{1}{2}} \gamma^2 e^{\phi+2\alpha}\beta \\ +[2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \gamma e^{-\phi})\end{array} & \begin{array}{c}(\gamma^2 e^{\phi+2\alpha}\beta^2 \\ +[2]\gamma e^{-\phi}\beta \\ +e^{-(3\phi+2\alpha)})\end{array} \end{pmatrix}. \tag{4.4}$$

Now, using relations (2.6), (3.4)–(3.8), we get

$$\mathcal{T}^{(1,\frac{1}{2})} = \xi \begin{pmatrix} a^2 & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} ab & \lambda^{-1} b^2 \\ [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} ac & ad + Q^{-1}\lambda^{-1} bc & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \lambda^{-1} bd \\ \lambda c^2 & [2]^{\frac{1}{2}} Q^{-\frac{1}{2}} \lambda cd & d^2 \end{pmatrix}. \tag{4.5}$$

Note that, in deriving (4.5) from (4.4), relations (3.8) are used in the Heisenberg–Weyl form

$$\begin{array}{llll} e^{\alpha}\beta = Q\lambda^{-1}\beta e^{\alpha} & e^{\alpha}\gamma = Q\lambda\gamma e^{\alpha} & e^{\phi}\beta = \lambda\beta e^{\phi} & e^{\phi}\gamma = \lambda^{-1}\gamma e^{\phi} \\ e^{\alpha}e^{\phi} = e^{\phi}e^{\alpha} & \beta\gamma = \gamma\beta\,. & & \end{array} \tag{4.6}$$

In the limit $p = q$, or $Q = q$ and $\lambda = 1$, $\mathcal{D}$ becomes a central element of $\mathcal{A}$ and $GL_{p,q}(2) \longrightarrow GL_q(2)$; further, choosing $\mathcal{D} = 1$ (or $\xi = 1$) leads to the quantum group $SL_q(2)$. In these cases, i.e. for $GL_q(2)$ and $SL_q(2)$, with $\lambda = 1$, z drops out of the picture and hence we may simply denote the $(2j+1)$-dimensional representation matrix as $\mathcal{T}^{(j)}$ which is obtained by taking $\lambda = 1$ in $\mathcal{T}^{(j,\frac{1}{2})}$. Then, for $SL_q(2)$ the matrix elements $\{\mathcal{T}^{(1)}_{mk} \mid m, k = 1, 0, -1\}$ of $\mathcal{T}^{(1)}$ in (4.5), with $\lambda = 1$ and $\xi = 1$, are seen to coincide with the spin-1 representation functions $\{d^1_{mk} \mid m, k = 1, 0, -1\}$ given by Nomura [10] (note: our q is Nomura's $q^{-\frac{1}{2}}$).

For $SU_q(2)$ one has to take into account further relations among $\{a, b, c, d\}$ due to the requirement of the existence of an involutional antihomomorphic $*$-operation satisfying

$$T^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = T^{-1} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \tag{4.7}$$

so that

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \tag{4.8}$$

with $q \in \mathbb{R}\backslash\{0\}$ necessarily. It may be noted that the unitarity condition (4.7), or $T^*T = TT^* = \mathbb{1}$, requires the relations

$$aa^* + q^2c^*c = a^*a + c^*c = 1 \tag{4.9}$$

to be satisfied, besides the commutation relations (2.2).

Let us now generalize the above result. For $\Gamma^{(j,\frac{1}{2})}$, the matrix elements are

$$\left.\begin{aligned} \hat{J}_{\pm,mk} &= Q^{\mp m+\frac{1}{2}}\{[j \pm m][j+1\mp m]\}^{\frac{1}{2}}\delta_{m,k\pm1} \\ \hat{J}_{0,mk} &= m\delta_{mk} \\ \hat{Z}_{mk} &= \tfrac{1}{2}\delta_{mk} \end{aligned}\right\} \qquad m,k = j, j-1, \ldots, -(j-1), -j\,. \tag{4.10}$$

Substituting this representation (4.10) in (3.9), it is found, after considerable algebraic manipulation, that one can write

$$\begin{aligned} \mathcal{T}_{mk}^{(j,\frac{1}{2})} &= \xi^{2j-1}Q^{-\frac{1}{2}(m-k)(2j-m+k)}\lambda^{-\frac{1}{2}(m-k)(2j-m-k-1)}\{[j+m]![j-m]![j+k]![j-k]!\}^{\frac{1}{2}} \\ &\quad \times\sum_s Q^{-s(2j-m+k-s)}\lambda^{-s(m-k+s)}\frac{a^{j+k-s}}{[j+k-s]!}\frac{b^{m-k+s}}{[m-k+s]!}\frac{c^s}{[s]!}\frac{d^{j-m-s}}{[j-m-s]!} \end{aligned} \tag{4.11}$$

where s runs from $\max(0, k-m)$ to $\min(j+k, j-m)$. In the limit $\lambda = 1$, $Q = q$ and $\xi = 1$, corresponding to $SL_q(2)$ (or $SU_q(2)$), the above expression (4.11) for $\mathcal{T}_{mk}^{(j)}$ coincides with Nomura's expression [10] for the quantum d-function d_{mk}^j (with our q replaced by Nomura's $q^{-\frac{1}{2}}$ as already noted). Nomura [10] has also noted the RTT relation (3.17) for the representation matrix $\mathcal{T}^{(j)}$ of $SU_q(2)$. When ξ is not taken specifically to be unity, the above matrices $\{\mathcal{T}^{(j)} \mid j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ (4.11), with $Q = q$ and $\lambda = 1$, provide the representations of $GL_q(2)$ (see [4] which gives these representations with an equivalent expression for the r.h.s. of (4.11)). The r.h.s. of (4.11) can be rearranged in several ways and so various equivalent expressions exist for $\mathcal{T}_{mk}^{(j)}$ in terms of different q-special functions for the cases of $SL_q(2)$ and $SU_q(2)$ (see [6, 7, 9, 11, 10]).

When $z \neq \frac{1}{2}$, the $\Gamma^{(j,z)}$-representation of $U_{p,q}(gl(2))$ is given by

$$\left.\begin{aligned} \hat{J}_{\pm,mk} &= \lambda^{z-\frac{1}{2}}Q^{\mp m+\frac{1}{2}}\{[j \pm m][j+1\mp m]\}^{\frac{1}{2}}\delta_{m,k\pm1} \\ \hat{J}_{0,mk} &= m\delta_{mk} \\ \hat{Z}_{mk} &= z\delta_{mk} \end{aligned}\right\}$$

$$m,k = j, j-1, \ldots, -(j-1), -j\,. \tag{4.12}$$

Substituting this representation (4.12) in (3.9) one gets

$$\mathcal{T}_{mk}^{(j,z)} = \lambda^{(m-k)(z-\frac{1}{2})}\xi^{1-2z}\mathcal{T}_{mk}^{(j,\frac{1}{2})} \qquad m,k = j, j-1, \ldots, -(j-1), -j\,. \tag{4.13}$$

As can be seen from (4.13), for generic z, the one- and two-dimensional representations are respectively given by

$$\mathcal{T}^{(0,z)} = \xi^{-2z} = \mathcal{D}^z \tag{4.14}$$

and

$$\mathcal{T}^{(\frac{1}{2},z)} = \mathcal{D}^{z-\frac{1}{2}} \begin{pmatrix} a & \lambda^{z-\frac{1}{2}} b \\ \lambda^{-(z-\frac{1}{2})} c & d \end{pmatrix}. \tag{4.15}$$

As has already been noted at the end of section 3, it can be seen that $\mathcal{T}^{(\frac{1}{2},z)}$ corresponds to the fundamental T-matrix of $GL_{p',q'}(2)$ with $p' = p^{z+\frac{1}{2}}/q^{z-\frac{1}{2}}$ and $q' = q^{z+\frac{1}{2}}/p^{z-\frac{1}{2}}$.

5. The quantum group $U_{\overline{q},q}(2)$

It is not possible to have $SU_{p,q}(2)$ with $p \neq q$ and for $SU_q(2)$ it is necessary that $q \in \mathbb{R}\backslash\{0\}$. However, we can have $U_{p,q}(2)$ with $p = \overline{q}$ (complex conjugate of q); we can have $U_{\overline{q},q}(2)$ for any $q \in \mathbb{C}\backslash\mathbb{R}$. For $q \in \mathbb{R}\backslash\{0\}$ one gets $U_q(2)$ of which $SU_q(2)$ is the special case corresponding to unit value for the quantum determinant. To see the features of $U_{\overline{q},q}(2)$ and $U_q(2)$ one has to study the consequences of imposing the unitarity condition on $GL_{p,q}(2)$ (see [28] for some useful details in this regard).

The fundamental T-matrix of $U_{\overline{q},q}(2)$, for $q = |q|\mathrm{e}^{\mathrm{i}\theta}$, is given by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -\overline{q}\,\mathcal{D}c^* \\ c & \mathcal{D}a^* \end{pmatrix} = \begin{pmatrix} a & -qc^*\mathcal{D} \\ c & a^*\mathcal{D} \end{pmatrix} \tag{5.1}$$

with

$$\begin{array}{lllll} ac = \overline{q}ca & a\mathcal{D} = \mathcal{D}a & ac^* = qc^*a & \mathcal{D}c^* = \mathrm{e}^{2\mathrm{i}\theta}c^*\mathcal{D} & cc^* = c^*c \\ \mathcal{D}^*\mathcal{D} = \mathcal{D}\mathcal{D}^* = 1 & aa^* + |q|^2c^*c = 1 & a^*a + c^*c = 1 & & \end{array} \tag{5.2}$$

and their $*$-conjugate relations, satisfied ($q^* = \overline{q}$). Note that

$$TT^* = T^*T = \mathbb{1} \tag{5.3}$$

in view of the relations (5.2), as required, and $\mathcal{D}$ may be representated as $\mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi^* = \varphi$.

If $q \in \mathbb{R}\backslash\{0\}$ the above equations (i.e. (5.1)–(5.3)) hold with $\theta = 0$ and $\mathcal{D}$ is a central element with values on the unit circle in $\mathbb{C}$: one gets $U_q(2)$. If the value of $\mathcal{D}$ is fixed to be unity $U_q(2) \longrightarrow SU_q(2)$. It is obvious that the representation matrices of $U_{\overline{q},q}(2)$ and $U_q(2)$ are given by equation (4.11) with relations (5.1) between $\{a, b, c, d\}$ taken into account.

6. Clebsch–Gordan coefficients for $U_{p,q}(gl(2))$ and $GL_{p,q}(2)$

Let $\mathcal{C}$ and $\mathcal{C}'$ be the Clebsch–Gordan matrices (CGMs) such that $\mathcal{C}^{-1}\Delta\mathcal{C}$ and $\mathcal{C}'^{-1}\Delta'\mathcal{C}'$ are reduced representations corresponding to the coproduct Δ in (1.2) and the opposite coproduct. From the relation (2.12) it is clear that $\mathcal{C}' = R\mathcal{C}$, where R is the R-matrix obtained from the universal $\mathcal{R}$ by substituting the corresponding irreducible representations involved in the coproduct (see [29] for a detailed discussion on the relation between $\mathcal{C}$s and R). Now, let us make the following observation on the CGMs for $U_{p,q}(gl(2))$ (or $U_{Q,\lambda}(gl(2))$): from (2.15) it is easy to see that

$$\mathcal{C}_{Q,\lambda} = F\mathcal{C}_{Q,\lambda=1} \qquad \mathcal{C}'_{Q,\lambda} = F^{-1}\mathcal{C}'_{Q,\lambda=1}\,. \tag{6.1}$$

Writing explicitly, one has the expressions for the (p, q) (or (Q, λ)) CGCs as

$$\begin{aligned} \langle j_1z_1m_1, j_2z_2m_2|jzm\rangle &= \lambda^{m_1z_2-z_1m_2}\delta_{z,z_1+z_2}\langle j_1m_1, j_2m_2|jm\rangle_Q \\ \langle j_1z_1m_1, j_2z_2m_2|jzm\rangle' &= \lambda^{z_1m_2-m_1z_2}\delta_{z,z_1+z_2}\langle j_1m_1, j_2m_2|jm\rangle'_Q \end{aligned} \tag{6.2}$$

where $\{\langle j_1m_1, j_2m_2|jm\rangle_Q\}$ and $\{\langle j_1m_1, j_2m_2|jm\rangle'_Q\}$ are the Q-CGCs of $U_Q(gl(2))$ corresponding respectively to the coproducts $\Delta_{Q,\lambda=1}$ and $\Delta'_{Q,\lambda=1}$ (see (2.11)).

It is particularly interesting to consider the coproducts for $U_{\overline{q},q}(u(2))$ with $q \in \mathbb{C}\backslash\mathbb{R}$. In this case, $Q = |q|$ and $\lambda = \mathrm{e}^{-\mathrm{i}\theta}$. Hence, the coproduct (2.11), with Hermitian Z and J_0, also preserves the Hermiticity property of the pair (J_+, J_-), i.e. $J_\pm^\dagger = J_\mp$. In the physical context this implies an addition of q-angular momenta of two particles, (1) and (2), according to the rule

$$\begin{aligned} &\Delta(J_\pm) = J_\pm(1)|q|^{-J_0(2)}\mathrm{e}^{\mp\mathrm{i}\theta Z(2)} + |q|^{J_0(1)}\mathrm{e}^{\pm\mathrm{i}\theta Z(1)}J_\pm(2) \\ &\Delta(J_0) = J_0(1) + J_0(2) \qquad \Delta(Z) = Z(1) + Z(2) \end{aligned} \tag{6.3}$$

with some 'phases' which may somehow be irremovable. In fact, one can even have $|q| = 1$, i.e. a modified addition rule for ordinary angular momenta with a new additive 'phase' quantum number. This aspect of the quantum algebra $U_{\overline{q},q}(u(2))$ may be worth probing further, particularly in view of the interest in physical applications of $U_{p,q}(u(2))$ (see [30]).

Let us now look at the direct product representations of the quantum group $GL_{p,q}(2)$. From (3.16) it follows that

$$\begin{aligned} &\mathcal{T}_1\mathcal{T}_2 = \mathcal{E}xp_{Q^{-2}}\{\gamma\Delta(\hat{J}_-)\}\exp\{\alpha(\Delta(\hat{J}_0) + \Delta(\hat{Z}))\}\mathcal{E}xp_{Q^2}\{\beta\Delta(\hat{J}_+)\} \\ &\mathcal{T}_2\mathcal{T}_1 = \mathcal{E}xp_{Q^{-2}}\{\gamma\Delta'(\hat{J}_-)\}\exp\{\alpha(\Delta'(\hat{J}_0) + \Delta'(\hat{Z}))\}\mathcal{E}xp_{Q^2}\{\beta\Delta'(\hat{J}_+)\} \end{aligned} \tag{6.4}$$

where Δ is given in (3.3) and the representations involved in the coproducts are the relevant representations $\{\Gamma^{(j,\frac{1}{2})}\}$. It is particularly easy to verify these relations in the case when Δ corresponds to the direct product of two identical $\Gamma^{(\frac{1}{2},\frac{1}{2})}$-representations. It is obvious that $\mathcal{C}^{-1}\mathcal{T}_1\mathcal{T}_2\mathcal{C}$ and $\mathcal{C}'^{-1}\mathcal{T}_2\mathcal{T}_1\mathcal{C}'$ will be in reduced forms. Since, for $U_{p,q}(gl(2))$, $\Delta^{(j_1,z_1)\otimes(j_2,z_2)} \longrightarrow \sum_{j'=|j_1-j_2|}^{j_1+j_2} \oplus\Gamma^{(j',z_1+z_2)}$, the direct product representation $\mathcal{T}_1\mathcal{T}_2 = (\mathcal{T}^{(j_1,z_1)} \otimes I^{(j_2,z_2)})(I^{(j_1,z_1)} \otimes \mathcal{T}^{(j_2,z_2)})$ (or $\mathcal{T}_2\mathcal{T}_1$ corresponding to the opposite coproduct), where $I^{(j,z)}$ is the $(2j + 1)$-dimensional identity matrix corresponding to the unity in the (j, z)-representation, will be reduced to the direct sum of the representation matrices $\{\mathcal{T}^{(j,z_1+z_2)}|j = |j_1 - j_2|, ..., j_1 + j_2\}$ and the corresponding CGCs are given by (6.2).

As an interesting example, let us consider the direct products of the representations labelled by $(0, z - \frac{1}{2})$ and $(j, \frac{1}{2})$ for both $U_{p,q}(gl(2))$ and $GL_{p,q}(2)$; note that, in general, the (j, z)-representation is defined by (4.12) (or (3.1)). Using the coproduct (3.3) and its opposite, the corresponding direct product representations of $U_{p,q}(gl(2))$ are given by, with $(j_1, z_1) = (0, z - \frac{1}{2})$ and $(j_2, z_2) = (j, \frac{1}{2})$,

$$\left.\begin{aligned} &\Delta(\hat{J}_+)_{mk} = Q^{-m+\frac{1}{2}}\{[j+m][j+1-m]\}^{\frac{1}{2}}\delta_{m,k+1} \\ &\Delta(\hat{J}_-)_{mk} = \lambda^{2z-1}Q^{m+\frac{1}{2}}\{[j-m][j+1+m]\}^{\frac{1}{2}}\delta_{m,k-1} \\ &\Delta(\hat{J}_0)_{mk} = m\delta_{mk} \\ &\Delta(\hat{Z})_{mk} = z\delta_{mk} \end{aligned}\right\} \\ m, k = j, j-1, \ldots, -(j-1), -j \tag{6.5}$$

and

$$\left.\begin{aligned} &\Delta'(\hat{J}_+)_{mk} = \lambda^{2z-1}Q^{-m+\frac{1}{2}}\{[j+m][j+1-m]\}^{\frac{1}{2}}\delta_{m,k+1} \\ &\Delta'(\hat{J}_-)_{mk} = Q^{m+\frac{1}{2}}\{[j-m][j+1+m]\}^{\frac{1}{2}}\delta_{m,k-1} \\ &\Delta'(\hat{J}_0)_{mk} = m\delta_{mk} \\ &\Delta'(\hat{Z})_{mk} = z\delta_{mk} \end{aligned}\right\} \\ m, k = j, j-1, \ldots, -(j-1), -j\,. \tag{6.6}$$

Using these representations in (6.4) above, it is seen that the corresponding direct product representations of $GL_{p,q}(2)$ are given by

$$\mathcal{T}_1\mathcal{T}_2 = \mathcal{D}^{z-\frac{1}{2}}\mathcal{T}^{(j,\frac{1}{2})} \qquad \mathcal{T}_2\mathcal{T}_1 = \mathcal{T}^{(j,\frac{1}{2})}\mathcal{D}^{z-\frac{1}{2}}. \tag{6.7}$$

It is easy to verify that the direct product representations of $U_{p,q}(gl(2))$ given by (6.5) and (6.6) can be 'reduced' to (or made equivalent to, in this case) the defining representation (4.12) using the CGCs obtained from (6.2) by taking the Q-CGMs to be identity matrices. The same CGMs are seen to 'reduce' also the direct product representations of $GL_{p,q}(2)$ given by (6.7) to the defining representation (4.13). It may also be noted that the direct product representations of $GL_{p,q}(2)$ given by (6.7) satisfy the RTT relation (3.17) with a $(2j+1)$-dimensional 'R-matrix' with elements $\{R_{mk} = \lambda^{(2z-1)m}\delta_{mk} | m, k = j, (j-1), \ldots, -(j-1), -j\}$ in accordance with (2.16).

7. Conclusion

Before closing, we may mention a few related points.

The converse of the exponential map, namely the passage $GL_{p,q}(2) \longrightarrow U_{p,q}(gl(2))$ using the representation of $GL_{p,q}(2)$ close to identity, for infinitesimal values of the group parameters $\{\alpha, \beta, \gamma, \delta\}$, follows by writing $\mathcal{T} \approx 1 + \gamma\hat{J}_- + \alpha(\hat{J}_0 + \hat{Z}) + \delta(\hat{J}_0 - \hat{Z}) + \beta\hat{J}_+$. Thus, if one can obtain the representations of $GL_{p,q}(2)$ by some method directly, then the representations of its infinitesimal generators $\{\hat{J}_0, \hat{J}_\pm, \hat{Z}\}$ forming the dual algebra $U_{p,q}(gl(2))$ can be derived. This is how Finkelstein [4] obtains the relationship $GL_q(2) \longrightarrow U_q(gl(2))$, independent of [1], using the theory of invariants to derive an equivalent form of the representation matrix (4.11) for the case $p = q$.

Finkelstein's analysis of the representations of $GL_q(2)$ [4] is motivated by the construction of a $GL_q(2)$ Yang–Mills theory in which one would regard the non-Abelian group parameters $\{\alpha(x), \beta(x), \gamma(x), \delta(x)\}$ (the coproduct rules of which specify the group multiplication law as pointed out in [3]) as space-time fields. Akulov *et al* [27] have considered the differential calculus of the group parameter space for $SL_q(2)$ and studied a related field-theory model. The problem of realization of the group parameters as dependent on continuous classical variables (like space-time) has been addressed recently by Volovich [31] at the level of the variables $\{a, b, c, d\}$ (see also [28]). Let us observe an example of such a realization based on the relations (3.7) and (3.8). Using the well known Bogoliubov transformation of a pair of commuting sets of boson operators, we can write

$$\alpha = (\rho - \theta)\psi_1^\dagger(x)\psi_1(x) + (\rho + \theta)\psi_2^\dagger(x)\psi_2(x)$$

$$\beta = \psi_1^\dagger(x) \qquad \gamma = \psi_2^\dagger(x)$$

$$\delta = (\rho + \theta)\psi_1^\dagger(x)\psi_1(x) + (\rho - \theta)\psi_2^\dagger(x)\psi_2(x)$$

with $\psi_1(x) = (\cosh x)a_1 - (\sinh x)a_2^\dagger$, $\psi_2(x) = (\cosh x)a_2 - (\sinh x)a_1^\dagger$, $x \in \mathbb{R}$, $[a_1, a_1^\dagger] = 1$, $[a_2, a_2^\dagger] = 1$, $[a_1, a_2] = 0$, $[a_1, a_2^\dagger] = 0$. In the context of building gauge theories based on quantum groups related to $GL(2)$ it is interesting to observe that in the case of $GL_{p,q}(2)$ the two-dimensional vector spaces carrying the fundamental representation ($\mathcal{T}$) have commuting components if $p = 1$, just like the Hilbert space of a two-level ordinary quantum mechanical system. Besides the interpretation of the matrix elements of the representations of $SU_q(2)$ as wavefunctions of quantum symmetric tops [10], generalization of quantum dynamics based on the representations of $SU_q(2)$ has also been considered [32]. Since having one more parameter would provide more flexibility in model building, we believe that the study of representations of $GL_{p,q}(2)$ should prove useful. Recently, generalization of the exponential map for the quantum supergroup $GL_{p,q}(1|1)$ has also been obtained [33].

There are several approaches to quantum groups (or quantum matrix pseudogroups) and quantum algebras (or quantized universal enveloping algebras) which are in duality with the quantum groups in the Hopf algebraic sense (see, e.g., [34, 35] for reviews of the subject). In the case of the quantum group $GL_{p,q}(2)$ [12–15] the structure of its dual Hopf algebra $U_{p,q}(gl(2))$ is known from the studies by Schirrmacher *et al* [16], using the non-commutative differential calculus approach, and by Dobrev [17, 18, 25] using the approach of Sudbery [36]. The recent analysis [1] of the duality relations between Lie bialgebras, with particular reference to $A(GL_{p,q}(2))$ and $U_{p,q}(gl(2))$, has led to a generalization of the well known exponential relationship between a classical Lie group and its Lie algebra and the explicit form of such an exponential map has been obtained between the quantum algebra $U_{p,q}(gl(2))$ and the corresponding quantum group $GL_{p,q}(2)$. This relationship is given abstractly in terms of a universal $\mathcal{T}$-matrix, involving both the group parameters of $GL_{p,q}(2)$ and the generators of $U_{p,q}(gl(2))$, and for particular representations of $U_{p,q}(gl(2))$ this universal $\mathcal{T}$-matrix gives the representations of $GL_{p,q}(2)$. Using this Fronsdal–Galindo formalism we have derived explicitly the finite-dimensional representations of $GL_{p,q}(2)$, by exponentiating directly the well known $(2j+1)$-dimensional irreducible representations of $U_{p,q}(gl(2))$, and the earlier results on the representations of $GL_q(2)$, $SL_q(2)$ and $SU_q(2)$ are found to be special cases in the appropriate limits. We have also derived the CGCs for the quantum algebra $U_{p,q}(gl(2))$ and noted that the same CGCs can be used for the Clebsch–Gordan reduction of the direct product representations of the quantum group $GL_{p,q}(2)$.

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